Short Papers

On the Evaluation of Modal Coupling Coefficients by Contour Integrals

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Abstract—**The coupling coefficients between two waveguide modes or between a waveguide and a Floquet mode can be written in terms of line integrals on the boundary of the smaller waveguide. However, some of these integrals give rise to indeterminate forms when the cutoff frequencies of the two modes coincide, thus making these expressions useless from a numerical point of view. In this paper, alternative line-integral expressions are derived, which remove the indeterminacy and are applicable also when the cutoff frequencies are very close or even coincident.**

***Index Terms*—Modal analysis, modal coupling coefficients, periodic structures, waveguides.**

I. Introduction

Many numerical methods for the analysis of electromagnetic structures (e.g., waveguide discontinuities [1]–[3], boxed monolithic microwave integrated circuits (MMICs) [4], [5], and frequency selective surfaces [6], [7]) require the calculation of modal coupling coefficients, represented by one of the following integrals:

$$\mathcal{I}_1 = \int_S \nabla\psi \cdot \nabla\mathcal{F}\, dS \tag{1}$$

$$\mathcal{I}_2 = \int_S \nabla\phi \cdot \nabla\mathcal{F}\, dS \tag{2}$$

$$\mathcal{I}_3 = \int_S \nabla\psi \cdot \nabla\mathcal{F} \times \vec{u}_z\, dS \tag{3}$$

$$\mathcal{I}_4 = \int_S \nabla\phi \cdot \nabla\mathcal{F} \times \vec{u}_z\, dS \tag{4}$$

where ψ and ϕ, defined in the domain S (see Fig. 1), are eigenfunctions of the Helmholtz equation with the Dirichelet and Neumann boundary conditions, respectively; $\mathcal{F}$, defined in the domain Ω including S, is an eigenfunction of the Helmholtz equation with Dirichelet, Neumann, or periodic boundary condition, dependent on the particular application [1]–[7]. S and Ω are arbitrarily shaped, with the obvious limitation on Ω in cases where $\mathcal{F}$ satisfies a periodic boundary condition (Floquet mode). Since the derivation to follow can be performed regardless to the boundary condition satisfied by $\mathcal{F}$, the same symbol is used to represent the function in all cases.

By using Green's identities and the properties of ψ, ϕ, and $\mathcal{F}$, integrals (1)–(4) can be transformed from surface to line integrals, as shown in [5], [6], [8], [9]

$$\mathcal{I}_1 = \frac{k^2}{k^2-\kappa^2}\int_{\partial S} \frac{\partial\psi}{\partial n}\mathcal{F}\, d\ell \tag{5}$$

$$\mathcal{I}_2 = \frac{\kappa^2}{\kappa^2-k^2}\int_{\partial S} \phi\frac{\partial\mathcal{F}}{\partial n}\, d\ell \tag{6}$$

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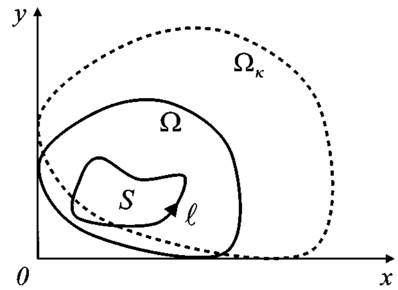

Fig. 1. Geometry considered in the calculation of the coupling integrals: the eigenfunctions ψ or ϕ are defined in the domain S, and $\mathcal{F}$ is defined in Ω. The eigenfunction $\mathcal{F}_\kappa$, introduced in Section II, is defined in the expanded domain Ω_κ.

$$\mathcal{I}_3 = 0 \tag{7}$$

$$\mathcal{I}_4 = \int_{\partial S} \phi\frac{\partial\mathcal{F}}{\partial\ell} d\ell \tag{8}$$

where κ is the eigenvalue associated to ψ or ϕ, and k is the eigenvalue associated to $\mathcal{F}$. The use of (5)–(8) is particularly convenient in cases where one or both eigenfunctions are calculated numerically.

Coupling coefficients $\mathcal{I}_1$ and $\mathcal{I}_2$ are finite also for $\kappa = k$, and this implies that (5) and (6) are indeterminate forms (0/0) in this particular case. Actually, due to numerical approximations in the calculation of the eigenfunctions, integrals appearing in (5) and (6) are not exactly zero when $\kappa = k$, and this gives rise to large errors when the eigenvalues are very close. Such an event is far from being rare, when a large number of eigenfunctions are considered.

In this paper, we derive alternative line-integral expressions for $\mathcal{I}_1$ and $\mathcal{I}_2$, which remove the indeterminacy of (5) and (6), thus preventing the said troubles.

II. Removing the Indeterminacy

The eigenfunction $\mathcal{F}$ and the eigenvalue k satisfy the Helmholtz equation. Let us consider a polar coordinate system (r, φ), where the origin of the system can be chosen arbitrarily. Introducing the normalized radius $R = kr$, the Helmholtz equation can be written as

$$\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial\mathcal{F}}{\partial R} + \frac{1}{R^2}\frac{\partial^2\mathcal{F}}{\partial\varphi^2} + \mathcal{F} = 0$$

which evidences that

$$\mathcal{F}(r,\varphi) = f(kr, \varphi). \tag{9}$$

It is evident that the domain Ω_κ of the function

$$\mathcal{F}_\kappa(r,\varphi) = f(\kappa r, \varphi) \tag{10}$$

differs from the domain Ω of $\mathcal{F}$ for a contraction ($\kappa > k$) or an expansion ($\kappa < k$) around the origin (Fig. 1). It is also evident that $\mathcal{F}_\kappa$ satisfies the Helmholtz equation

$$\nabla^2\mathcal{F}_\kappa + \kappa^2\mathcal{F}_\kappa = 0$$

and the same boundary condition as $\mathcal{F}$, over the boundary of Ω_κ. Both the functions $\mathcal{F}$ and $\mathcal{F}_\kappa$ can be continued analytically outside their respective domains, so that $\mathcal{F}_\kappa$ is defined in the whole domain Ω, even if Ω is not totally included in Ω_κ, like in Fig. 1. In particular $\mathcal{F}_\kappa$ is defined over the boundary ∂S.

We observe that, due to the boundary condition $\psi = 0$ over ∂S, we have

$$\begin{aligned}\int_{\partial S} \frac{\partial \psi}{\partial n}\mathcal{F}_\kappa d\ell &= \int_{\partial S}\left(\mathcal{F}_\kappa \frac{\partial \psi}{\partial n} - \psi\frac{\partial \mathcal{F}_\kappa}{\partial n}\right)d\ell \\ &= \int_S (\mathcal{F}_\kappa \nabla^2\psi - \psi\nabla^2\mathcal{F}_\kappa)\, dS \\ &= 0\end{aligned}$$

because both ψ and $\mathcal{F}_\kappa$ are eigenfunctions of the Helmholtz equation, with the same eigenvalue κ. By a similar reasoning we can show that, due to the boundary condition $\partial\phi/\partial n = 0$ over ∂S, we also have

$$\int_{\partial S} \phi \frac{\partial \mathcal{F}_\kappa}{\partial n} d\ell = 0.$$

As a consequence, the integrals in (5) and (8) remain unchanged if $\mathcal{F}$ is replaced with $\mathcal{F} - \mathcal{F}_\kappa$. Then, introducing the function

$$\mathcal{F}' = k\frac{\mathcal{F} - \mathcal{F}_\kappa}{k - \kappa} \tag{11}$$

we can write

$$\mathcal{I}_1 = \frac{k}{k+\kappa}\int_{\partial S}\mathcal{F}'\frac{\partial\psi}{\partial n}\, d\ell \tag{12}$$

$$\mathcal{I}_2 = -\frac{\kappa^2}{k(k+\kappa)}\int_{\partial S}\phi\frac{\partial \mathcal{F}'}{\partial n}d\ell. \tag{13}$$

From (10) and (9) we infer

$$\mathcal{F}_\kappa(r,\,\varphi) = \mathcal{F}\left(\frac{\kappa}{k}r, \varphi\right)$$

so that

$$\mathcal{F}'(r,\,\varphi) = \frac{\mathcal{F}\left(\frac{\kappa}{k}\,r,\,\varphi\right) - \mathcal{F}(r,\,\varphi)}{\frac{\kappa}{k} - 1}. \tag{14}$$

Solutions of the Helmholtz equation are continuously differentiable, so that we can write

$$\mathcal{F}\left(\frac{\kappa}{k}\,r,\,\varphi\right) = \mathcal{F}(r,\,\varphi) + \sum_{m=1}^{\infty}\frac{1}{m!}\frac{\partial^m\mathcal{F}(r,\,\varphi)}{\partial r^m}\left(\frac{\kappa}{k}\,r - r\right)^m$$

when

$$\mathcal{F}' = \sum_{m=1}^{\infty}\frac{r^m}{m!}\frac{\partial^m\mathcal{F}(r,\varphi)}{\partial r^m}\left(\frac{\kappa}{k} - 1\right)^{m-1}. \tag{15}$$

In cases where the eigenfunction $\mathcal{F}$ is known analytically (e.g., $\mathcal{F}$ is the potential of a mode of a rectangular or circular waveguide) the function $\mathcal{F}'$ can be evaluated with good accuracy by using (14), even if k is very close to κ. In particular, in the critical case $k = \kappa$ the function $\mathcal{F}'$ is given by the first term of the power expansion

$$\mathcal{F}' = r\frac{\partial\mathcal{F}(r,\,\varphi)}{\partial r} \qquad (\text{case } k = \kappa). \tag{16}$$

In cases where $\mathcal{F}$ and its first and second radial derivatives are known numerically, the expression

$$\mathcal{F}' \approx r\frac{\partial\mathcal{F}(r,\,\varphi)}{\partial r} + \frac{r^2}{2}\frac{\partial^2\mathcal{F}(r,\,\varphi)}{\partial r^2}\left(\frac{\kappa}{k} - 1\right) \tag{17}$$

can be used to obtain a good approximation of $\mathcal{F}'$, when k is close to κ.

In conclusion, the use of (12) and (13) removes any problem deriving from the indeterminacy of expressions (5) and (6) at $k = \kappa$.

III. Analytical Formulas for Particular Cases

In some particular cases, where the eigenfunction $\mathcal{F}$ is known analytically, expressions of $\mathcal{F}'$ were derived, which are valid for any value of κ/k. In the following, expressions are reported in the cases of rectangular waveguide modes and of Floquet modes.

A. Rectangular Waveguide Modes

In the case of a TM mode of a rectangular waveguide, $\mathcal{F}$ is given by

$$\mathcal{F}(x,\,y) = A\sin(k_x x)\sin(k_y y) \tag{18}$$

where A is the normalization factor and $k_x^2 + k_y^2 = k^2$. Specific expressions of k_x and k_y are well known [10]. By using (14), it can be derived

$$\begin{aligned}\mathcal{F}'(x,\,y) = A\frac{\sigma}{2}\sin\left(\frac{\kappa+k}{2k}\,\sigma\right)\operatorname{sinc}\left(\frac{\kappa-k}{2\pi k}\,\sigma\right)\\ -A\frac{\delta}{2}\sin\left(\frac{\kappa+k}{2k}\,\delta\right)\operatorname{sinc}\left(\frac{\kappa-k}{2\pi k}\,\delta\right)\end{aligned} \tag{19}$$

where

$$\sigma = k_x x + k_y y$$
$$\delta = k_x x - k_y y$$

and $\operatorname{sinc}(\alpha) = \sin(\pi\alpha)/\pi\alpha$.

In the case of a TE mode, $\mathcal{F}$ is given by

$$\mathcal{F}(x,\,y) = B\cos(k_x x)\cos(k_y y) \tag{20}$$

where B is the normalization factor. By using (14), it can be derived that

$$\begin{aligned}\mathcal{F}'(x,\,y) = -B\frac{\sigma}{2}\sin\left(\frac{\kappa+k}{2k}\,\sigma\right)\operatorname{sinc}\left(\frac{\kappa-k}{2\pi k}\,\sigma\right)\\ -B\frac{\delta}{2}\sin\left(\frac{\kappa+k}{2k}\,\delta\right)\operatorname{sinc}\left(\frac{\kappa-k}{2\pi k}\,\delta\right).\end{aligned} \tag{21}$$

B. Floquet Modes

In the case of a Floquet mode, $\mathcal{F}$ is given by

$$\mathcal{F}(x,\,y) = Ce^{j(k_x x + k_y y)} \tag{22}$$

where C is the normalization factor and $k_x^2 + k_y^2 = k^2$. Specific expressions of k_x and k_y are given in [11]. By using (14), it can be derived that

$$\mathcal{F}'(x,\,y) = j\,C\,\sigma\, e^{j((\kappa+k)/2k)\sigma}\operatorname{sinc}\left(\frac{\kappa-k}{2\pi k}\,\sigma\right). \tag{23}$$

IV. Numerical Examples

In this section we discuss through examples the numerical problems connected with the use of (5) and (6), and how both the exact and the approximated expressions derived in this paper permit to overcome these problems.

The first case we consider is the calculation of the coupling coefficients between two rectangular waveguides. When the eigenvalues and eigenfunctions are known analytically, the calculation of the coupling coefficients by using (5) and (6) fails only when κ and k are exactly coincident. When $\kappa = k$, expressions (12) and (13) can be used in conjunction with (16).

Conversely, when the eigensolutions of the Helmholtz equation are obtained numerically, the use of (5) and (6) fails when the two eigenvalues are close. In fact, in expressions (5) and (6), the denominator

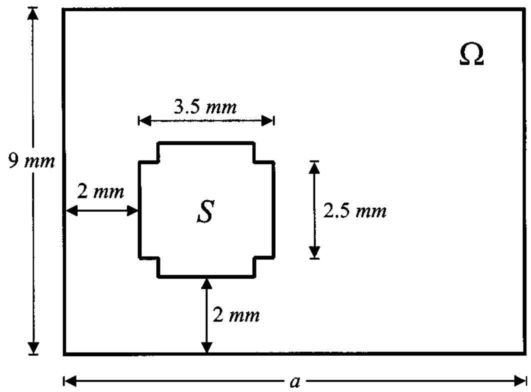

Fig. 2. Cross section of the cross-shaped waveguide S and of the rectangular waveguide Ω.

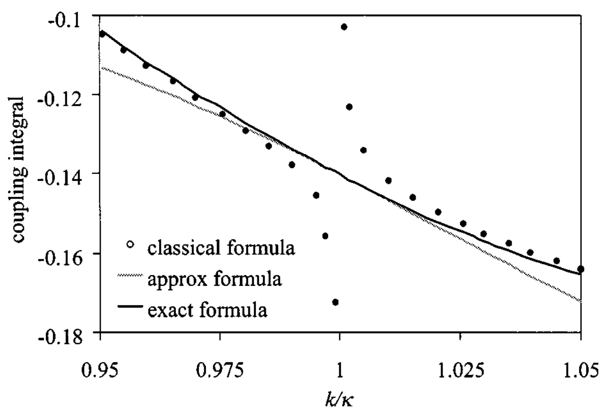

Fig. 3. Coupling integral $\mathcal{I}_1$ between the first TM mode of the cross-shaped waveguide and the TM_{42} of the rectangular waveguide: the dots are obtained by (5), the black line by (12) and (19), and the gray line by (12) and (17).

vanishes when $\kappa = k$, whereas the numerator is not exactly zero when $\kappa = k$. This phenomenon is the origin of large errors in the calculation of the coupling coefficients, as highlighted in the following example.

We consider the calculation of the coupling integral $\mathcal{I}_1$ between the TM modes of a rectangular and a cross-shaped waveguide (see Fig. 2). The normalization $\int_S \psi^2\, dS = \kappa^2$ and $\int_\Omega \mathcal{F}^2\, dS = k^2$ is assumed. In this example, the eigenfunction ψ of the cross-shaped waveguide was obtained numerically by using the boundary integral-resonant mode expansion (BI-RME) method [12].

Since the larger waveguide is rectangular, three different expressions can be used for the calculation of the coupling integral $\mathcal{I}_1$: the *classical* expression (5), which exhibits the indeterminacy when $k = \kappa$; the *novel* formula (12) with the analytical expression (19) of $\mathcal{F}'$, which can be evaluated without numerical problems for any k and κ; the *novel* formula (12) with the approximated expression (19) of $\mathcal{F}'$, which provides accurate results if $k \approx \kappa$.

Fig. 3 shows the value of the coupling integral between the first TM-mode eigenfunction of the cross-shaped waveguide ($\kappa = 1305\ \mathrm{m}^{-1}$) and the mode TM_{42} of the rectangular waveguide, when varying the dimension a of the rectangular domain. This dimension ranges from 12.96 to 10.66 mm, so that the ratio k/κ ranges from 0.95 to 1.05. In Fig. 3, the dots denote the integral calculated by (5), the black line refers to the exact calculation obtained by using (12) and (19), whereas the gray line refers to the approximated calculation obtained by using (12) and (17).

It is worth observing that no problem is encountered at any value of the ratio k/κ when using (12) and (19). Conversely, the use of (5) provides the proper value of the integral if k and κ are different enough (say, more than $\pm 2.5\%$), while a significant discrepancy from the exact value is found when $k \approx \kappa$. Finally, the use of (12) and (17) leads to a good approximation of the integral if $k \approx \kappa$ [i.e., exactly where (5) fails]. Therefore, it is a valid solution when the analytical expression of $\mathcal{F}'$ is not available, provided that radial derivatives appearing in (17) can be calculated numerically with a sufficient accuracy.

V. Conclusion

Alternative line-integral expressions were derived for the calculation of modal coupling integrals, which are valid also when the cutoff frequencies of the two eigenfunctions coincide. Coupling coefficients were obtained in the form of a power expansion. In particular cases, closed forms were found. An example demonstrates the effectiveness of these formulas.

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